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RANDOM MEASURES

THESIS

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RANDOM MEASURES

THESIS

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Abstract

A random measure may be thought of as a random set function which is almost surely a measure. The general objective of this investigation is to extend the theory of random measures and specifically to characterize homogeneous random measures - roughly, random measures whose behavior conforms in a certain way to that of a fixed measure. Some results obtained by Ryll-Nardzewski for point processes on the real line are extended and the Laplace functional, a useful device for characterizing random measures, is introduced. Completely random measures, infinitely divisible random measures, and stationary random measures are characterized. Homogeneous random measures are introduced with examples and interpretations. A general characterization theorem for homogeneous random measures is proved. Finally, several applications of the theory of random measures are given.

Random Measures

I. Introduction

Historical Perspective

A random measure can be considered as a random function on a sigma-algebra of subsets of some space which is almost surely a measure on that space. The earliest investigations of random measures concerned themselves with point processes, an important class of random measures taking on nonnegative integer values. Ryll-Nardzewski (Ref 28) established a measure theoretic framework for Khintchine's results (Ref 16) in the area of stationary point processes on the real line. Moyal (Ref 25) investigated random measures in the context of stochastic population processes laying down the foundations of a general theory. Similarly, Harris (Ref 14) studied a particular class of stochastic population processes known as branching processes - a mathematical model of the development of a population whose members reproduce and die, subject to laws of chance. Goldman (Ref 11) studied various transformations of point processes on R^n such as clustering, deleting points, superposition, and random translations. Emphasis was placed on the asymptotic behavior of well-distributed point processes under iterations of the operations. In a following paper, Goldman (Ref 12) investigated the generalization of the concept of infinitely divisible point processes on R^n in terms of superposition. Agnew (Ref 2) characterized the behavior of uniform and stationary point processes under transformations with particular emphasis on super-

position and decomposition.

Bochner (Ref 3) and LeCam (Ref 19) each investigated finitely additive set functions in the late 1940's and, in a later work, Bochner (Ref 4:137-142) characterized a special class of random functions in terms of their characteristic functionals. Harris (Ref 15) also investigated random set functions having appropriately smooth realizations. However, it was not until Kingman (Ref 17) introduced a special class of random measures which he called completely random that random measures were explicitly treated. Subsequently, Lee (Ref 21) investigated infintely divisible random measures (specifically point processes) characterizing them in terms of the Laplace transforms of their finite dimensional distributions. In a later paper Lee (Ref 22) presented several examples of these point processes, obtaining specialized results for certain types of point processes such as the generalized Poisson process and the bulk Poisson process. Finally, Mecke (Ref 23) elegantly treated stationary random measures on a locally compact Abelian group, characterizing them in terms of Palm measure.

Scope of the Investigation

The general objective of this investigation is to extend the theory of random measures and to indicate some applications of random measures. A specific objective is to characterize homogeneous random measures.

Overview

In Chapter II the notion of a random measure is formalized.

Several results which clarify the properties of random measures are obtained, and some results given by Ryll-Nardzewski (Ref 28) for point processes on the real line are extended. Also, the Laplace functional (a useful device for investigating random measures) is introduced. Finally, some interesting classes of random measures which have previously been investigated are detailed.

Homogeneous random measures are introduced in Chapter III. Examples of homogeneous random measures are given with interpretation where appropriate and the relationship of homogeneous random measures to the other classes of random measures is also discussed. Finally, a general characterization theorem for homogeneous random measures is proved.

In Chapter IV some examples which illustrate applications of the theory of random measures are given.

II. Random Measures

Random Set Functions

Let $(X, \mathcal{B})$ be a measurable space with $\mathcal{B}$ containing at least every singleton set (i.e., $\{\{x\} : x \in X\} \subset \mathcal{B}$). Given an underlying probability space $(\Omega, \mathcal{A}, P)$, a real random set function on $(X, \mathcal{B})$ can be defined as a stochastic process on $\mathcal{B} \times \Omega$ (i.e., a real-valued function on $\mathcal{B} \times \Omega$ whose $\mathcal{B}$-sections are $\mathcal{A}$-measurable). A real random set function m induces a unique probability measure $\tilde{P}$ on the product space $(R, \mathcal{R})^{\mathcal{B}} = (R^{\mathcal{B}}, \mathcal{R}^{\mathcal{B}})$, where $(R, \mathcal{R})$ is the extended real line and its Borel sets, via the measurable transformation T defined by

$$T(w) = m(\cdot, w) \tag{1}$$

(i.e., $\tilde{P} = PT^{-1}$). By the Kolmogorov extension theorem (Ref 14:53), $\tilde{P}$ is characterized uniquely in terms of its finite-dimensional distributions. In order to investigate distribution properties, it suffices to characterize a real random set function as a probability measure on $(R, \mathcal{R})^{\mathcal{B}}$. Of course, this approach ignores the subtle aspects of equivalent stochastic processes.

Attention in the literature has been focused on random set functions which are almost surely measures primarily because a great body of theory exists for measures which one might use to advantage. We similarly restrict our attention to random measures, but we note that more general random set functions, such as those which are almost surely superadditive, have application and are worthy of study.

Random Measures

Let $M \subset R^{\mathcal{B}}$ be the set of measures on $(X, \mathcal{B})$, and let $\mathfrak{M} = M \cap \mathcal{R}^{\mathcal{B}}$

DEFINITION 1. A random measure on $(X, \mathcal{B})$ is a probability measure on $(M, \mathfrak{M})$.

A random measure is uniquely determined by its finite dimensional distributions and these distributions must satisfy certain consistency conditions. Given $B_1, \ldots, B_n \in \mathcal{B}$ and $C_1, \ldots, C_r \in \mathcal{B}$ disjoint such that $B_i = \bigcup_{k \in K_i} C_k$ for $i = 1, \ldots, n$, let $I_k = \{i: k \in K_i\}$. Then, we must have

$$E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i)\right\}\right] = E\left[\exp\left\{-\Sigma_{k=1}^{r}(\Sigma_{i \in I_k} t_i) m(C_k)\right\}\right] \quad (2)$$

for all $t_1, \ldots, t_n \geq 0$. It follows that all finite-dimensional distributions are specified once those corresponding to disjoint sets are specified. Furthermore, $P\{m(\emptyset) = 0\} = 1$ so that

$$E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i) - tm(\emptyset)\right\}\right] = E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i)\right\}\right] \quad (3)$$

for all $B_1, \ldots, B_n \in \mathcal{B}$ and $t_1, \ldots, t_n, t \geq 0$. Finally, we have the monotone continuity condition

$$E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i) - tm(C_k)\right\}\right]$$
$$\downarrow E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i) - tm(C)\right\}\right] \quad (4)$$

whenever $B_1, \ldots, B_n, C, C_1, \ldots \in \mathcal{B}$, $t_1, \ldots, t_n, t \geq 0$ and $C_k \uparrow C$.

Whether or not such a set of consistent finite-dimensional distributions always extends to a unique random measure seems to be an unanswered question in general. Harris (Ref 14:55) has proved the existence of a unique extension in a special case, and Bochner has proved a generalized extension to $(S, \mathcal{J})$, where S is the set of finitely additive set functions on $\mathcal{B}$ and $\mathcal{J} = S \cap \mathcal{R}^{\mathcal{B}}$, whenever (1) holds. If P is the unique extension to $(R^{\mathcal{B}}, \mathcal{R}^{\mathcal{B}})$ and $P^*(M) = 1$ (P^* is the outer measure associated with P), then P induces a unique probability measure $\tilde{P}$ on $(M, \mathcal{M})$ called the trace of P on M (Ref 27:19). Thus, if the consistency conditions imply $P^*(M) = 1$, then the unique extension follows.

LEMMA 2. If P is a random measure on $(X, \mathcal{B})$, then the set function η defined on $\mathcal{B}$ by

$$\eta(\cdot) = E\left[m(\cdot)\right] = \int m(\cdot)P(dm) \tag{5}$$

is in M.

PROOF. We have that $\eta(\emptyset) = E\left[m(\emptyset)\right] = 0$, and

$$\eta(\cup B_i) = E\left[m(\cup B_i)\right] = E\left[\Sigma m(B_i)\right] = \Sigma E\left[m(B_i)\right] = \Sigma\eta(B_i) \tag{6}$$

for disjoint $\{B_i\} \subset \mathcal{B}$ by monotone convergence or by Fubini's theorem (Ref 13:148).

Let F be the set of nonnegative $\mathcal{B}$-measurable functions on X. Let $\tilde{F} \subset F$ be the subset of simple functions of the form $f = \Sigma_{i=1}^{n} t_i 1_{B_i}$ where $B_1, \ldots, B_n \in \mathcal{B}$ are disjoint and $t_1, \ldots, t_n \geq 0$, and where 1_{B_i}

is the indicator function of the set B_i defined by

$$1_{B_i}(x) = \begin{cases} 1 & x \in B_i \\ & \text{if} \\ 0 & x \notin B_i \end{cases} \tag{7}$$

If $f \in F$, then there exists a nondecreasing sequence $\{f_n\} \subset \tilde{F}$ such that $f_n \uparrow f$, and $\int f_n dm \uparrow \int f dm$ for all $m \in M$ (Ref 13:85).

LEMMA 3. If $f \in F$, then y defined on M by

$$y(m) = \int f dm \tag{8}$$

is $\mathcal{M}$-measurable.

PROOF. If $f \in \tilde{F}$, then

$$\int f dm = \int \Sigma_{i=1}^{n} t_i 1_{B_i} dm = \Sigma_{i=1}^{n} t_i m(B_i) \tag{9}$$

which is clearly measurable. Now, if $\{f_n\} \subset \tilde{F}$ and $f_n \uparrow f$, then $\int f_n dm \uparrow \int f dm$ so that y is the limit of a nondecreasing sequence of measurable functions and is thus measurable (Ref 13:84), which was to be proved.

THEOREM 4. If $f \in F$ and P is a random measure on $(X, \mathcal{B})$, then

$$E\left[\int f dm\right] = \int\int f dm P(dm) = \int f d\eta \tag{10}$$

PROOF. If $f \in \tilde{F}$, we have

$$E\left[\int f dm\right] = E\left[\Sigma_{i=1}^{n} t_i m(B_i)\right]$$

$$= \Sigma_{i=1}^{n} t_i E\left[m(B_i)\right]$$

$$= \Sigma_{i=1}^{n} t_i \eta(B_i) = \int f d\eta \qquad (11)$$

and the assertion follows by taking limits since $\{f_n\} \subset \tilde{F}$ and $f_n \uparrow f$ implies that $\int f_n dm \uparrow \int f dm$ for $m \in M$ and since $\eta \in M$ we have $\int f_n d\eta \uparrow \int f d\eta$. $E\left[\int f_n dm\right] \uparrow E\left[\int f dm\right]$ from monotone convergence.

THEOREM 5. If P is a random measure on $(X, \mathcal{B})$ and $B, C \in \mathcal{B}$ are disjoint, then

$$P\left\{m(B \cup C) > 0\right\} \leq P\left\{m(B) > 0\right\} + P\left\{m(C) > 0\right\} \qquad (12)$$

PROOF.

$$P\left\{m(B \cup C) > 0\right\} = P\left\{m(B) + m(C) > 0\right\}$$

$$= P\left\{\left\{m(B) > 0\right\} \cup \left\{m(C) > 0\right\}\right\}$$

$$\leq P\left\{m(B) > 0\right\} + P\left\{m(C) > 0\right\} \qquad (13)$$

LEMMA 6. $E\left[\exp\left\{-tm(B)\right\}\right] \downarrow P\left\{m(B) = 0\right\}$ as $t \uparrow \infty$.

PROOF. As $t \uparrow \infty$, for each $m \in M$ we have

$$\exp\left\{-tm(B)\right\} \downarrow 1_{\{m(B) = 0\}}(m) \qquad (14)$$

hence

$$E\left[\exp\left\{-tm(B)\right\}\right] \downarrow E\left[1_{\{m(B)=0\}}\right] = P\left\{m(B)=0\right\} \tag{15}$$

DEFINITION 7. If P_1 and P_2 are random measures, the superposition or convolution of P_1 and P_2, denoted by $P_1 * P_2$, is the random measure defined by

$$(P_1 * P_2)(A) = \iint 1_A(m_1+m_2)\; P_1(dm_1)\; P_2(dm_2) \tag{16}$$

Ryll-Nardzewski's Function

In this section we extend some results given by Ryll-Nardzewski (Ref 28) for point processes on the real line.

THEOREM 8. If η is σ-finite, then there exists a function Q on $\mathcal{M} \times X$ such that for each $A \in \mathcal{M}$, $Q(A, \cdot\,) \in F$ is uniquely determined (except perhaps on a set of η-measure zero) by the equation

$$\int_B Q(A,x)\; \eta\,(dx) = \int_A m(B)\; P\,(dm) \tag{17}$$

which holds for every $B \in \mathcal{B}$. Q has the following properties

(1) $Q(\emptyset,\cdot) = 0$ and $Q(M,\cdot) = 1$ $\qquad [\eta]^*$ (18)

(2) $A_1 \subset A_2 \Rightarrow Q(A_1,\cdot) \leq Q(A_2,\cdot)$ $\qquad [\eta]$ (19)

(3) $\{A_i\} \subset \mathcal{M}$ disjoint $\Rightarrow Q(\cup A_i,\cdot) = \Sigma Q(A_i,\cdot)$ $\qquad [\eta]$ (20)

---

* (Read almost everywhere η) denotes that the statement immediately preceeding it is true except perhaps on a set whose η -measure is zero.

PROOF. $\eta_A(\cdot) = \int_A m(\cdot)P(dm)$ defines a measure on $\mathcal{B}$ which is absolutely continuous with respect to η. Hence, the Radon-Nikodym theorem (Ref 13:128) assures the existence and uniqueness $[\eta]$ of $Q(A, \cdot)$. (1) follows by uniqueness of $Q(A, \cdot)$ and (2) is easily proved by contradiction.

$\eta_{\cup A_i} = \Sigma\eta_{A_i}$ so that

$$\int_B Q(\cup A_i,x)\eta(dx) = \Sigma\int_B Q(A_i,x)\eta(dx)$$

$$= \int_B \Sigma Q(A_i,x)\eta(dx) \tag{21}$$

for all $B \in \mathcal{B}$ and (3) follows by uniqueness.

THEOREM 9. $Q(\{m(C) = 0\},\cdot) = 0\ [\eta]$ on $C \in \mathcal{B}$.

PROOF.

$$\int_B Q(\{m(C) = 0\},x)\eta(dx) = \int_{\{m(C) = 0\}} m(B)P(dm)$$

$$= \int_{\{m(C) = 0\}} m(B \cap C^c)P(dm)$$

$$\leq \eta(B \cap C^c)$$

$$\leq \int_B 1_{C^c}d\eta \tag{22}$$

for all $B \in \mathcal{B}$ which implies that

$$Q(\{m(C) = 0\},\cdot) \leq 1_{C^c} \quad [\eta] \tag{23}$$

from which the theorem follows.

The following lemma is computationally useful.

LEMMA 10. Suppose that B, $C \in \mathcal{B}$ and $\eta(B) < \infty$. Then,

$$\lim_{t\to\infty} \lim_{u\to 0} -\frac{\partial}{\partial u} E\left[\exp\left\{-um(B)-tm(C)\right\}\right]$$

$$= \int_{\{m(C) = 0\}} m(B)P(dm) \tag{24}$$

PROOF.

$$\frac{\partial}{\partial u} E\left[\exp\left\{-um(B)-tm(C)\right\}\right] = -E\left[m(B)\exp\left\{-um(B)-tm(C)\right\}\right] \tag{25}$$

Now, $\eta(B) < \infty \Rightarrow m(B) < \infty$ almost surely which implies that as $u \downarrow 0$

$$m(B)\exp\left\{-um(B)-tm(C)\right\} \uparrow m(B)\exp\left\{-tm(C)\right\} \tag{26}$$

almost surely so that

$$E\left[m(B)\exp\left\{-um(B)-tm(C)\right\}\right] \uparrow E\left[m(B)\exp\left\{-tm(C)\right\}\right] \tag{27}$$

Furthermore,

$$m(B)\exp\left\{-tm(C)\right\} \downarrow m(B)1_{\{m(C) = 0\}} \tag{28}$$

almost surely as $t \uparrow \infty$ and

$$E\left[m(B)\exp\left\{-tm(C)\right\}\right] \leq \eta(B) < \infty \tag{29}$$

It follows that

$$E\left[m(B)\exp\left\{-tm(C)\right\}\right] \downarrow E\left[m(B)1_{\{m(C) = 0\}}\right] \tag{30}$$

<u>Laplace Functionals</u>

In this section we introduce the natural analog for random measures of the Laplace transform of a probability measure on

$(R^+, \mathcal{R}^+)$ ($R^+ = [0, \infty]$ and $\mathcal{R}^+ = R^+ \cap \mathcal{R}$). Harris (Ref 14:56) applies the name moment generating functional to what we call the Laplace functional. Mecke (Ref 24) has also used the term Laplace functional.

DEFINITION 11. The functional Φ defined on F by

$$\Phi(f) = E\left[\exp\left\{-\int f dm\right\}\right] = \int \exp\left\{-\int f dm\right\} P(dm) \qquad (31)$$

is called the Laplace functional of the random measure P.

THEOREM 12. The Laplace functional of a random measure is uniquely determined and has the following properties

(1) $0 \leq \Phi(f) \leq 1$ for all $f \in F$ (32)

(2) $f_1 \leq f_2 \Rightarrow \Phi(f_1) \geq \Phi(f_2)$ (33)

(3) $f_n \uparrow f \in F \Rightarrow \Phi(f_n) \downarrow \Phi(f)$ (34)

(4) If $f = \Sigma_{i=1}^{n} t_i 1_{B_i} \dot{\in} \tilde{F}$, $\Phi(f)$ is the joint Laplace transform of $(m(B_1), \ldots, m(B_n))$ evaluated at $(t_1, \ldots, t_n)$.

(5) $\Phi(tf)$ is the Laplace transform of the random variable $\int f dm$ evaluated at $t \geq 0$.

(6) $\Phi(f) \geq \exp\left\{-\int f d\eta\right\}$ (35)

PROOF. (1) and (2) are trivial. Now, $f_n \uparrow f$ implies $\int f_n dm \uparrow \int f dm$ for all $m \in M$ hence $\exp\left\{-\int f_n dm\right\} \downarrow \exp\left\{-\int f dm\right\}$ for all $m \in M$ which implies that

$$E\left[\exp\left\{-\int f_n dm\right\}\right] \downarrow E\left[\exp\left\{-\int f dm\right\}\right] \qquad (36)$$

and (3) follows. To prove (4) it suffices to note that for $f \in \tilde{F}$ we have

$$\begin{aligned}\Phi(f) &= \Phi(\Sigma t_i 1_{B_i}) \\ &= E\left[\exp\left\{-\int \Sigma t_i 1_{B_i} dm\right\}\right] \\ &= E\left[\exp\left\{-\Sigma t_i m(B_i)\right\}\right] \qquad (37)\end{aligned}$$

Since for $f \in F$, $\int f dm$ is a random variable (5) is obvious. Uniqueness follows from (3) and (4) and (6) is a consequence of Theorem 4 and Jensen's inequality (Ref 27:55).

THEOREM 13. If Φ_1 and Φ_2 are the Laplace functionals of P_1 and P_2 respectively, then the Laplace functional of $P_1 * P_2$ is,

$$\Phi = \Phi_1 \cdot \Phi_2 \qquad (38)$$

PROOF. The assertion follows from the fact that

$$\int u(m) P_1 * P_2(dm) = \iint u(m_1 + m_2) P_1(dm_1) P_2(dm_2) \qquad (39)$$

for any nonnegative $\mathcal{M}$-measurable function u, which can be proved without difficulty by the usual procedure of employing simple functions and then taking limits.

Completely Random Measures

In this section we present some results obtained by Kingman (Ref 17). In particular a representation theorem is presented for completely random measures satisfying a weak finiteness condition.

DEFINITION 14. A random measure P is said to be completely random if, for any finite collection $B_1, \ldots, B_n$ of disjoint members of $\mathcal{B}$, the random variables $m(B_1), \ldots, m(B_n)$ are independent.

THEOREM 15. Let P be a completely random measure. If η is nonatomic and if there exists a countable collection $\{B_n\} \subset \mathcal{B}$ such that $\cup B_n = X$ and $P\{m(B_n) < \infty\} > 0$ for all n, then there exist σ-finite measures $\beta, \pi_1, \pi_2, \ldots$ on $(X, \mathcal{B})$ and probability measures $p_\nu(x, \cdot)$ ($\nu = 1,2,\ldots;\ x \in X$) on $(R^+, \mathcal{R}^+)$ with $p_\nu(\cdot, E)$ $\mathcal{B}$-measurable for each $E \in \mathcal{R}^+$, such that the Laplace functional of P admits the representation

$$\Phi(f) = \exp\left\{-\int f d\beta - \Sigma_{\nu=1}^{\infty} \int (1 - p_\nu^*(x,f)) \pi_\nu(dx)\right\} \tag{40}$$

where

$$p_\nu^*(x,t) = \int_0^\infty \exp(-tz) p_\nu(x,dz) \tag{41}$$

PROOF. Kingman showed that

$$E\left[\exp\{-tm(B)\}\right] = \exp\left\{-t\beta(B) - \Sigma_{\nu=1}^{\infty} \int_B (1 - p_\nu^*(x,t)) \pi_\nu(dx)\right\} \tag{42}$$

but the complete randomness of P implies

$$\begin{aligned} \Phi(\Sigma_{i=1}^n t_i 1_{B_i}) &= E\left[\exp\{-\Sigma_{i=1}^n t_i m(B_i)\}\right] \\ &= \prod_{i=1}^n E\left[\exp\{-t_i m(B_i)\}\right] \\ &= \exp\{-\Sigma_{i=1}^n t_i \beta(B_i)\} \\ &\quad -\Sigma_{i=1}^n \Sigma_{\nu=1}^{\infty} \int_{B_i} (1 - p_\nu^*(x,t_i)) \pi_\nu(dx)\} \end{aligned} \tag{43}$$

Now,

$$\Sigma_{i=1}^{n}\Sigma_{\nu=1}^{\infty}\int_{B_i}(1-p_\nu^*(x,t_i))\Pi_\nu(dx)$$
$$= \Sigma_{\nu=1}^{\infty}\int\Sigma_{i=1}^{n}(1-p_\nu^*(x,t_i))1_{B_i}\Pi_\nu(dx)$$
$$= \Sigma_{\nu=1}^{\infty}\int(1-p_\nu^*(x,\Sigma_{i=1}^{n}t_i 1_{B_i}))\Pi_\nu(dx) \qquad (44)$$

which proves the assertion for simple functions and the result follows upon taking limits. We use here the convention $p_\nu^*(x,0) = 1$, even when $p_\nu^*(x,0^+) < 1$.

COROLLARY.

$$\eta(B) = \beta(B) + \Sigma_{\nu=1}^{\infty}\int_B\int z p_\nu(x,dz)\Pi_\nu(dx) \qquad (45)$$

The theorem implies that $P = Q*\Delta_\beta$ where Q has the Laplace functional

$$\Phi(f) = \exp\left\{-\Sigma_{\nu=1}^{\infty}\int(1-p_\nu^*(x,f))\Pi_\nu(dx)\right\} \qquad (46)$$

and

$$\Delta_\beta(A) = \begin{cases} 1 & \text{if } \beta\in A \\ 0 & \text{otherwise} \end{cases} \qquad (47)$$

DEFINITION 16. Let $\mu\in M$ be σ-finite and nonatomic and let h be the Laplace transform of a probability measure on $(0,\infty)$. A random measure with Laplace functional

$$\Phi(f) = \exp\left\{-\int(1-h(f))d\mu\right\} \qquad (48)$$

is called a Poisson process with intensity measure μ and mass distribution h.

In the simple Poisson point process, we have $h(t) = \exp(-t)$, the Laplace transform of a unit mass at one. It is clear that the Poisson process corresponds to Kingman's representation with $p_\nu^*(x,\cdot) = h(\cdot)$ for all x, $\pi_1 = \mu$, and $\beta, \pi_2, \pi_3, \ldots$ all zero measures.

We shall have more to say about the Poisson process in the next chapter.

Infinitely Divisible Random Measures

A random measure P is infinitely divisible if for any positive integer n,

$$P = \overset{n}{*} P_n \tag{49}$$

for some random measure P_n. An example of an infinitely divisible point process is the Poisson process with intensity measure μ which, for any n, is the n-fold superposition of a Poisson process with the same mass distribution and intensity measure μ/n.

Combining the results of Lee and Mecke, we have the following theorem.

THEOREM 17. A random measure P is infinitely divisible if and only if its Laplace functional admits a representation.

$$\Phi(f) = \exp\left\{-\int f\,d\beta - \int \left[1-\exp\left(-\int f\,dm\right)\right] \Lambda(dm)\right\} \tag{50}$$

where $\beta \in M$ and Λ is a measure on $(M, \mathcal{M})$.

COROLLARY.

$$\eta(B) = \beta(B) + \int m(B)\Lambda(dm) \qquad (51)$$

The theorem indicates that $P = Q*\Delta_\beta$ where Q has Laplace functional

$$\Phi(f) = \exp\left\{-\int\left[1-\exp(-\int f dm)\right]\Lambda(dm)\right\} \qquad (52)$$

In the case of a Poisson process, we have

$$\int\left[1-\exp(-\int f dm)\right]\Lambda(dm) = \int(1-h(f))d\mu$$
$$= \int\int(1-\exp(-\lambda f))\,\xi(d\lambda)d\mu \qquad (53)$$

where

$$h(t) = \int_0^\infty \exp(-\lambda t)\,\xi(d\lambda) \qquad (54)$$

which implies that Λ is concentrated on measures of the form $\lambda\delta_x$ where

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \qquad (55)$$

and

$$\Lambda\left\{\lambda\delta_x : \lambda \in I, x \in B\right\} = \xi(I)\mu(B) \qquad (56)$$

<u>Stationary Random Measures</u>

Suppose that X is a commutative group with respect to an operation conveniently denoted as addition; i.e., there exists an operation

$+ : X \times X \to X$ such that $x + y = y + x \in X$ whenever $x, y \in X$; there exists an identity element $0 \in X$ such that $x + 0 = x$ for all $x \in X$; and for each $x \in X$, there exists a unique inverse element $-x \in X$ such that $x + (-x) = 0$. Let $\{U_x : x \in X\}$ be the group of translation operators on X defined by $U_x y = y + x$. We assume that $\mathcal{B}$ is closed under translation (i.e., $B \in \mathcal{B} \Rightarrow U_x B \in \mathcal{B}$, $\forall\, x \in X$). Let $\{T_x : x \in X\}$ be the group of translation operators on M defined by

$$(T_x m)(B) = m(U_{-x} B) \tag{57}$$

A measure $\mu \in M$ is translation invariant if $T_x \mu = \mu$ for all $x \in X$.

DEFINITION 18. A random measure P on $(X, \mathcal{B})$ is stationary if

$$P(T_x A) = P(A) \tag{58}$$

for all $A \in \mathcal{M}$ and $x \in X$.

THEOREM 19. P is stationary if, and only if,

$$\int u(T_x m) P(dm) = \int u(m) P(dm) \tag{59}$$

for every nonnegative $\mathcal{M}$-measurable function u.

PROOF. If P is stationary then

$$\begin{aligned} \int u(T_x m) P(dm) &= \int u(m) P T_x^{-1}(dm) \\ &= \int u(m) P T_{-x}(dm) \\ &= \int u(m) P(dm) \end{aligned} \tag{60}$$

Conversely, suppose that

$$\int u(T_x m)P(dm) = \int u(m)P(dm) \tag{61}$$

for all u. Then,

$$\begin{aligned} P(T_x A) &= \int 1_{T_x A}(m)P(dm) \\ &= \int 1_A(T_{-x} m)P(dm) \\ &= \int 1_A(m)P(dm) \\ &= P(A) \end{aligned} \tag{62}$$

so that P is stationary.

Let $\left\{ V_x : x \in X \right\}$ be the group of translation operators defined on F by

$$(V_x f)(y) = f(U_{-x} y) = f(y-x) \tag{63}$$

THEOREM 20. P is stationary if, and only if,

$$\Phi(V_x f) = \Phi(f) \tag{64}$$

for all $f \in F$ and $x \in X$ where Φ is the Laplace functional of P.

PROOF. If P is stationary, then

$$\begin{aligned} \Phi(V_x f) &= \int \exp\left\{ -\int V_x f dm \right\} P(dm) \\ &= \int \exp\left\{ -\int f dT_{-x} m \right\} P(dm) \\ &= \int \exp\left\{ -\int f dm \right\} P(dm) \\ &= \Phi(f) \end{aligned} \tag{65}$$

by the previous theorem with $u(m) = \exp\left\{ - \int f dm \right\}$.

Conversely, suppose that $\Phi(V_x f) = \Phi(f)$. Put $f = \Sigma_{i=1}^{n} t_i 1_{B_i}$ and we have

$$V_x f = \Sigma_{i=1}^{n} t_i 1_{U_x B_i} \tag{66}$$

so that

$$E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i)\right\}\right] = E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(U_x B_i)\right\}\right] \tag{67}$$

It follows that the finite dimensional distributions of P are translation invariant and it follows by extension that P is stationary.

For a stationary random measure P we have the following ergodic theorem.

THEOREM 21. Fix $x \in X$, $B \in \mathfrak{G}$ and let $\mathcal{B} = \left\{ A \in \mathfrak{M} : T_x A = A \right\}$. If P is stationary and $\eta(B) < \infty$, then

$$\lim_{n \to \infty} \frac{1}{n+1} \left[\Sigma_{k=0}^{n} m(U_x^k B)\right] = E\left[m(B) \mid \mathcal{B}\right] \tag{68}$$

almost surely and in L^1.

PROOF. $m(U_x^k B) = (T_{-x}^k m)(B)$ and $P(T_{-x}^{-1} A) = P(T_x A) = P(A)$ for all $A \in \mathfrak{M}$ by stationarity. Hence, the assertion follows from Birkhoff's ergodic theorem (Ref 27:210).

THEOREM 22. If P is stationary, then η is translation invariant.

PROOF.

$$\begin{aligned}\eta(U_xB) &= \int m(U_xB)P(dm)\\ &= \int (T_{-x}m)(B)P(dm)\\ &= \int m(B)PT_x(dm)\\ &= \int m(B)P(dm)\\ &= \eta(B) \qquad (69)\end{aligned}$$

by stationarity.

THEOREM 23. Suppose that there exists a translation invariant probability measure $\mu \in M$. Then, if P is any random measure, the random measure P' defined by

$$P'(A) = \int P(T_xA)\mu(dx) \qquad (70)$$

is stationary.

PROOF. Using Fubini's theorem,

$$\begin{aligned}P'(T_yA) &= \int P(T_{y+x}A)\mu(dx)\\ &= \int 1_{T_{y+x}A}(m)P(dm)\mu(dx)\\ &= \int 1_{T_{y+x}A}(m)\mu(dx)P(dm)\\ &= \int 1_{T_xA}(m)T_y\mu(dx)P(dm)\\ &= \int 1_{T_xA}(m)\mu(dx)P(dm)\\ &= \int 1_{T_xA}(m)P(dm)\mu(dx)\\ &= \int P(T_xA)\mu(dx)\\ &= P'(A) \qquad (71)\end{aligned}$$

which was to be proved.

Finally, we give the following characterization for stationary random measures on a locally compact Abelian topological group X with Borel sets $\mathcal{B}$.

THEOREM 24. Let $\mu \in M$ be any probability measure that is not concentrated on any subgroup of X, and suppose that $P(T_x A)$ is a uniformly continuous function on X for each $A \in \mathcal{M}$. Then, P is stationary if, and only if,

$$P(A) = \int P(T_{-x}A)\mu(dx) \tag{72}$$

for all $A \in \mathcal{M}$ and $x \in X$.

PROOF. The necessity is obvious. T- --ove the sufficiency, fix A and put $f(x) = P(T_{-x}A)$. Then we have

$$f(y) = \int f(y-x)\mu(dx) \tag{73}$$

for all $y \in X$ and f must be constant on X by a theorem of Choquet and Deny (Ref 5).

REMARK 25. Mecke has obtained many beautiful characterizations for stationary random measures particularly in terms of Palm measure, which are not included here. The interested reader is referred to Mecke's paper (Ref 23) for additional results.

III. Homogeneous Random Measures

In this chapter we introduce the notion of a homogeneous random measure.* Some results which indicate interesting properties of homogeneous random measures are obtained and several examples plus interpretations are given. Characterizations are given for completely random and stationary homogeneous random measures. Finally, a general characterization theorem is proved.

Homogeneous Random Measures

DEFINITION 26. A random measure P is homogeneous with respect to a nonatomic, σ-finite measure $\mu \in M$ if its Laplace functional admits a representation

$$\Phi(f) = g\left[\int (1-h(f))d\mu\right] \tag{74}$$

where h is the Laplace transform of a probability distribution on $(0,\infty)$ and g is completely monotone on $(0,\mu(X))$ with $g(0^+) = 1$.

The following result is an immediate consequence of the definition.

$$E\left[\exp\left\{-\Sigma_{i=1}^{n} t_i m(B_i)\right\}\right] = \Phi(\Sigma_{i=1}^{n} t_i 1_{B_i})$$

$$= g\left[\int (1-h(\Sigma_{i=1}^{n} t_i 1_{B_i}))d\mu\right]$$

---

* Nawrotzki (Ref 26) has written a paper on homogeneous random point functions which we have been unable to obtain.

$$= g\left[\int \Sigma_{i=1}^{n} (1-h(t_i)) 1_{B_i} d\mu\right]$$

$$= g\left[\Sigma_{i=1}^{n} (1-h(t_i))\mu(B_i)\right] \tag{75}$$

Using this result, we have the following lemma.

LEMMA 27. If P is a random measure which is homogeneous with respect to μ then

$$P\left\{m(B) = 0\right\} = g(\mu(B)) \tag{76}$$

PROOF. By lemma 6

$$P\left\{m(B) = 0\right\} = \lim_{t \to \infty} E\left[\exp\left\{-tm(B)\right\}\right] \tag{77}$$

Using the result just obtained we have

$$P\left\{m(B) = 0\right\} = \lim_{t \to \infty} g\left[(1-h(t))\mu(B)\right] \tag{78}$$

and the desired result follows. Similarly we obtain another useful result.

LEMMA 28. Let P be a μ-homogeneous random measure and let $B \in \mathcal{B}$, then

$$\eta(B) = h'(0^+)g'(0^+)\mu(B) \tag{79}$$

PROOF. If $\mu(B) < \infty$, m(B) is almost surely finite, so that

$$\begin{aligned} \eta(B) &= \lim_{t \to \infty} -\frac{\partial}{\partial t} g\left[(1-h(t))\mu(B)\right] \\ &= \lim_{t \to \infty} g'\left[(1-h(t))\mu(B)\right]h'(t)\mu(B) \\ &= g'(0^+)h'(0^+)\mu(B) \end{aligned} \tag{80}$$

and the assertion follows from the σ-finiteness of μ.

The following interpretation is possible. Since h(t) is the Laplace transform of a probability distribution on $(0,\infty)$, $-h'(0^+)$ may be thought of as the expected mass at a point. We may consider $-g'(0^+)\mu(B)$ as the expected number of points in the set B. Hence, $\eta(B) = [-h'(0^+)][-g'(0^+)\mu(B)] = h'(0^+)g'(0^+)\mu(B)$ may be thought of as the expected mass in the set B.

THEOREM 29. Let P be μ-homogeneous and $0 < -g'(0^+)$, $-h'(0^+) < \infty$, then

$$Q(\{m(B) = 0\},x) = \begin{cases} 0 & x \in B \\ & \text{if} \qquad [\eta] \\ g'(\mu(B))/g'(0^+) & x \notin B \end{cases} \tag{81}$$

PROOF. By theorem 8 we have

$$\int_C Q(\{m(B) = 0\},x)\eta(dx) = \int_{\{m(B) = 0\}} m(C)P(dm) \tag{82}$$

Applying lemma 10 we obtain for $\eta(C)<\infty$, $B \cap C = \emptyset$,

$$\begin{aligned}\int_C Q(\{m(B)=0\},x)\eta(dx) &= \lim_{t\to\infty}\lim_{u\to 0} -\frac{\partial}{\partial u} g[(1-h(u))\mu(C) \\ &\qquad +(1-h(t))\mu(B)] \\ &= \lim_{t\to\infty}\lim_{u\to 0} g'[(1-h(u))\mu(C) \\ &\qquad +(1-h(t))\mu(B)]h'(u)\mu(C) \\ &= g'(\mu(B))h'(0^+)\mu(C) \end{aligned} \tag{83}$$

Using σ-finiteness, the equality holds whenever $B \cap C = \emptyset$.

Using lemma 10, theorem 9, and uniqueness, the assertion follows.

Examples of Homogeneous Random Measures

In this section we present several examples of homogeneous random measures.

EXAMPLE 30. Let $\mu(X) = 1$, and put $g(u) = (1 - u)^k$ and $h(t) = \exp(-t)$ so that

$$\begin{aligned} \Phi(f) &= g(\int(1-h(t))d\mu) \\ &= [1-\int(1-\exp(-f))d\mu]^k \end{aligned} \tag{84}$$

For $f = \sum t_i 1_{B_i}$ this yields

$$\begin{aligned} \Phi(f) &= E[\exp\{-\Sigma t_i m(B_i)\}] \\ &= [1-\Sigma\mu(B_i)+\Sigma\mu(B_i)\exp(-t_i)]^k \end{aligned} \tag{85}$$

which corresponds to picking k points independently from a space X, each according to the probability measure μ.

To generalize this result we substitute a general h obtaining

$$\Phi(f) = [1-\int(1-h(f))d\mu]^k \tag{86}$$

which yields for $f \in \tilde{F}$

$$E[\exp\{-\Sigma t_i m(B_i)\}] = [1-\Sigma\mu(B_i)+\Sigma\mu(B_i)h(t_i)]^k \tag{87}$$

As before, we select k points independently from X according to the probability measure μ, but we then assign to each point a random mass via the distribution whose Laplace transform is h.

EXAMPLE 31. Let $g(u) = \exp(-\lambda u)$ and $h(t) = \exp(-t)$. Then,

$$\Phi(f) = \exp\left\{ -\lambda\int(1-\exp(-f))d\mu\right\} \tag{88}$$

so that for $f \in \tilde{F}$ we have

$$\Phi(f) = \exp\left\{ -\lambda\Sigma\mu(B_i)(1-\exp(-t_i))\right\} \tag{89}$$

This is the familiar Poisson point process. Again we can generalize by substituting an arbitrary h to obtain

$$\Phi(f) = \exp\left\{ -\lambda\int(1-h(f))d\mu\right\} \tag{90}$$

which is a compound or bulk Poisson process with random masses assigned independently according to h.

The integer valued compound Poisson process puts $h(t) = f(\exp(-t))$ where f is a probability generating function on the positive integers.

The final example we present is the Mixed Poisson process.

EXAMPLE 32. If $\{P_\lambda\}$ is any family of random measures indexed over an index set Λ then the set function given by

$$P = \int_\Lambda \nu(d\lambda)P_\lambda \tag{91}$$

is again a random measure whose Laplace functional is

$$\Phi(f) = \int_\Lambda \nu(d\lambda)\Phi_\lambda(f) \tag{92}$$

where Φ_λ is the Laplace functional of the random measure P_λ.

If

$$\Phi_\lambda(f) = \exp\left\{ -\lambda\int(1-h(f))d\mu\right\} \tag{93}$$

and ν is a distribution on $[0,\infty)$, then

$$\Phi(f) = g(\int(1-h(f))d\mu) \tag{94}$$

where

$$g(u) = \int \exp(-\lambda u)\nu(d\lambda) \tag{95}$$

is the Laplace transform of ν. For example, if

$$g(u) = \beta/(\beta+u) \tag{96}$$

then

$$\Phi(f) = \beta/[\beta+\int(1-h(f))d\mu] \tag{97}$$

Clearly, if $\mu(X) = \infty$, any homogeneous random measure is mixed Poisson since g is necessarily the Laplace transform of a probability distribution on $[0,\infty)$.

INTERPRETATION 33. The following interpretation is possible. Let $0 < \mu(B) < \infty$ and put

$$\nu(C) = \mu(C \cap B)/\mu(B) \tag{98}$$

Then, for all f with $\{f > 0\} \subset B$,

$$\begin{aligned}\Phi(f) &= g[\int(1-h(f)d\mu] \\ &= g[\int_B(1-h(f))d\mu] \\ &= g[\mu(B)(1-\int h(f)d\nu)] \\ &= \sum_{n=0}^{\infty} p_n[\int h(f)d\nu]^n \\ &= \sum_{n=0}^{\infty} p_n[1-\int(1-h(f))d\nu]^n \end{aligned} \tag{99}$$

where p_n is the probability distribution on the nonnegative integers whose generating function is $g(\mu(B)(1-z))$. The interpretation is that n points are assigned to the set B with probability p_n, and then these points are selected independently from B according to the normalized measure ν with masses assigned via h. For example: if

$$g(u) = \beta/(\beta+u) \,, \tag{100}$$

then

$$\begin{aligned} g[\mu(B)(1-z)] &= \beta/[\beta+\mu(B)(1-z)] \\ &= \beta(\beta+\mu(B))^{-1}/[1-\{\mu(B)(\beta+\mu(B))^{-1}z\}] \end{aligned} \tag{101}$$

which is the generating function of the geometric distribution with

$$p_n = [1-\mu(B)(\beta+\mu(B))^{-1}][\mu(B)(\beta+\mu(B))^{-1}]^n \tag{102}$$

If $\mu(X)<\infty$, we can put B = X in the above. In this case, g need not be a Laplace transform.

A generalization of picking k points at random when μ is a probability measure would put

$$g(u) = (1-pu)^k \tag{103}$$

which yields

$$g(1-z) = (q+pz)^k \tag{104}$$

where $q = 1-p$ and

$$p_n = \binom{k}{n} p^n q^{k-n} \ ; \quad 0 \leqslant n \leqslant k \tag{105}$$

In the following section we give a necessary and sufficient condition for a μ-homogeneous random measure to be completely random.

<u>Completely Random Homogeneous Random Measures</u>

THEOREM 34. A μ-homogeneous random measure with $\mu(X) = \infty$ is completely random if, and only if, it is a Poisson process.

PROOF. The sufficiency is obvious. To prove the necessity we note that μ σ-finite and nonatomic implies that

$$g(u_1+u_2) = g(u_1)g(u_2) \tag{106}$$

for all u_1, $u_2 > 0$ (Ref 13:174) which implies that

$$g(u) = \exp(-\lambda u) \tag{107}$$

for some $\lambda \geqslant 0$ (Ref 1:183) and the necessity follows.

<u>Stationary Homogeneous Random Measures</u>

In this section we obtain necessary and sufficient conditions for a μ-homogeneous random measure to be stationary.

THEOREM 35. A μ-homogeneous random measure $0 < -h'(0^+)$, $-g'(0^+) < \infty$ is stationary if, and only if, μ is translation invariant.

PROOF. Suppose that μ is translation invariant then,

$$\begin{aligned}\Phi(V_x f) &= \exp\left\{ -\int(1-h(V_x f))d\mu\right\} \\ &= \exp\left\{ -\int(1-h(f)dT_{-x}\mu\right\}\end{aligned}$$

$$= \exp\left\{ -\int (1-h(f))d\mu \right\}$$
$$= \Phi(f) \qquad (108)$$

and the sufficiency follows from a previous result.

Conversely, if P is stationary, then η is translation invariant which implies that μ is translation invariant.

We conclude this chapter with a general characterization theorem for homogeneous random measures.

A Characterization Theorem

THEOREM 36. Let $(X, \mathcal{B}, \mu)$ be a σ-finite, nonatomic measure space with $\mu(X) = \infty$. Let P be a random measure on $(X, \mathcal{B})$. Suppose that all finite dimensional distributions of P for disjoint sets $B_1, \ldots, B_n \in \mathcal{B}$ with finite measure have the "homogeneity" property

$$E\left[\exp(-\Sigma_{i=1}^{n} t_i m(B_i))\right] = g_n(\mu(B_1), \ldots, \mu(B_n), t_1, \ldots, t_n) \qquad (109)$$

Put

$$g(u) = \lim_{t \to \infty} g_1(u,t) \qquad (110)$$

and suppose that g is the Laplace transform of a nontrivial (not concentrated at zero) probability distribution on $[0,\infty)$ with $g^{(k)}(u) = O(g'(u))$ as $u \downarrow 0$ for $k \geq 2$. Put $f_n = g^{-1} g_n$ and suppose that

$$f_n(u_1, \ldots, u_n, t_1, \ldots, t_n) = \Sigma_{i=1}^{n} f_1(u_i, t_i) \qquad (111)$$

Then, P is μ-homogeneous.

PROOF. From Halmos (Ref 13:174), we have the continuity property $0 \leq \alpha \leq \mu(B)$ implies that there exists a $C \subset B$ such that $\mu(C) = \alpha$ and $\mu(B) \leq \alpha$ implies that there exists a $C \supset B$ such that $\mu(C) = \alpha$. Now, $g(0^+) = 1$ implies that

$$P\left\{m(B_n) > 0\right\} \downarrow 0 \tag{112}$$

as $\mu(B_n) \downarrow 0$ and hence $m(B_n) \longrightarrow 0$ in probability as $\mu(B_n) \downarrow 0$. This implies that there is a subsequence $\left\{B_{n_j}\right\}$ such that $m(B_{n_j}) \longrightarrow 0$ almost surely (Ref 27:47). If $\left\{B_n\right\}$ is monotone decreasing and $\mu(B_n) \downarrow 0$, it follows that $m(B_n) \longrightarrow 0$ almost surely. From this and the continuity property from Halmos, it follows that g_n is continuous in each u_i. Furthermore, g_n has the additivity property (resulting from the additivity of μ)

$$g_n(u_1,\ldots,u_n,t,\ldots,t) = g_1(\Sigma_{i=1}^n u_i,t) \tag{113}$$

for $t > 0$. It follows that g_n maps into the domain of g^{-1} so that $f_n = g^{-1}g_n$ is well defined. Using (111) and the additivity property, we have

$$f_1(\Sigma_{i=1}^n u_i,t) = \Sigma_{i=1}^n f_1(u_i,t) \tag{114}$$

which for each $t > 0$ is a Cauchy equation with solution

$$f_1(u,t) = u\rho(t) \tag{115}$$

for some continuous, strictly increasing function ρ (Ref 1:34). Now

$f_1(u,t) \downarrow 0$ implies that $\rho(t) \downarrow 0$ as $t \downarrow 0$ and $f_1(u,t) \uparrow u$ implies that $\rho(t) \uparrow 1$ as $t \uparrow \infty$ and we have

$$g_n(u_1,\ldots,u_n,t_1,\ldots,t_n) = g(\Sigma_{i=1}^n u_i\,\rho(t_i)) \tag{116}$$

The composition $g \circ u\rho$ is the Laplace transform of a nontrivial probility distribution on $[0,\infty)$ for each $u > 0$. We need to show that ρ has derivatives of all orders and that ρ' is completely monotone. For $t > 0$

$$\lim_{\tau\to 0} \frac{\rho(t+\tau)-\rho(t)}{\tau} = \lim_{\tau\to 0} \frac{\dfrac{g(\rho(t+\tau))-g(\rho(t))}{\tau}}{\dfrac{g(\rho(t+\tau))-g(\rho(t))}{\rho(t+\tau)-\rho(t)}}$$

$$= \frac{(g\circ\rho)'(t)}{g'(t)} > 0 \tag{117}$$

Since $g' < 0$ on $(0,\infty)$ and both the numerator and denominator have derivatives of all orders, it follows that ρ has derivatives of all orders. We now invoke a theorem of Bochner (Ref 4:83) to prove that the derivatives of ρ oscillate in sign and hence that $\rho = 1-h$ where h is the Laplace transform of a probability distribution on $(0,\infty)$.

REMARK. The condition $g^{(k)}(u) = O(g'(u))$ is satisfied if $\left|g^{(k)}(0^+)\right| < \infty$ for all k, i.e., all moments are finite. We have included (111) as a hypothesis, however, it may be a necessary consequence of the invariance of g_n under permutations of (u_i,t_i) pairs.

IV. Applications

In this chapter we present several examples which illustrate areas in which the application of the theory of random measures might be useful.

An Application to Telephone Engineering

The following example is due to Fortet (Ref 10:84-85). As a telephone exchange has only a finite number of links, if too many people try to put in calls at the same time, there will be congestion; some calls will either be delayed or lost. In order to provide adequate customer service, the telephone engineer requires information such as the probability that a call arising at some specific time will find the exchange in a state such that the call is lost. Other information that might be required is the distribution (on the time axis) of the calls that are not lost, or the number of conversations held at some specific time t as a function of t.

One method of obtaining such information is to construct a mathematical model utilizing the theory of random measures that explicitly describes the processing of a call through the exchange. Formally, we assume that subscribers make calls at instants T_i which are considered as random instants and that the number of calls in the interval $[0, t)$ is $N(t)$. Let

$$m(t_1,t_2) = N(t_1) - N(t_2). \tag{109}$$

Then, clearly, the random set function m induced on the Borel sets of the time axis is a random measure. By assuming that this measure is stationary or completely random one is able to obtain additional information about the process. For example, by assuming that the random measure m is the sum of n independent random measures m_i (such as the number of out-of-town calls in a given time interval) and that each m_i is a stationary Poisson Process with parameter λ_i, Fortet obtained that the probability $P_i(t)$, that a call made at time t belonging to the i-th class of calls will be lost, is a constant P_i. Fortet was also able to obtain results which are quite difficult to obtain using classical procedures. For example, the expected number of calls belonging to a particular class and held at time t was quite easily obtained.

The next example, due to LeCam (Ref 20), is similar to the previous one in that the idea of a random measure - specifically, a point process - is central to the model utilized.

An Application to Conservation Studies

In order to assess the effects of constructing dams on the surrounding countryside a description of the random structure of stream flow is desirable. Such a description is possible only if one first starts with an adequate description of the random structure of rainfall. The model proposed by LeCam is essentially a random measure of the type known as a cluster process (Ref 12:19).

The process describing rainfall can be interpreted in the following manner. Points in the atmosphere are selected to be centers of

storms according to a Poisson process whose expectation is given by a "climatic" measure. A random mass is assigned to each point selected which determines the strength of the storm centered at that point. Continuing in this fashion centers of fronts are selected and given extent and velocity. Finally, centers of convective cells are chosen via another Poisson process and the amount of water precipitated by each cell is determined by a random mass assigned to each cell.

In this model the number of points chosen to be -- centers of storms, centers of fronts, and centers of convective cells are random measures.

The following example, due to Whittle (Ref 30) illustrates an advantage of working with random measures.

An Application to Agricultural Studies

Whittle formulated an agricultural model in terms of a random measure in order to evaluate the spatial covariance function of yield density. To accomplish the same objective using the classical approach would have been impractical if not impossible. By considering the yield of a plot of ground as a random measure and by empirically determining its variance he was able to determine the spatial covariance function in terms of the variance of the yield density.

Another area in which the application of the theory of random measures might be useful is illustrated by the following example.

An Application to Military Systems Analysis

A decision-maker is faced with the problem of defending an area consisting of a finite number of targets from ballistic missile

attack. Were his resources unlimited, the decision-maker's problem would be solved. It would only be necessary for him to allocate sufficient defensive systems to each target to ensure complete success against any attack. Typically, however, resources are limited to the extent that it is not feasible to defend every target. Hence, the decision-maker must choose the targets to be defended. To make this choice the decision-maker requires knewledge of the value placed on the targets by the enemy (i.e., the enemy's utility for the targets).

However, since the enemy is not a single entity, there is no "enemy utility function" per se. Instead, the enemy is a group of decision-makers which is constantly changing both in its composition and its objectives. The decision-maker is thus faced with the problem of "hitting a moving target."

One approach to this problem might be to use a point estimate of the enemy's utility for a set of targets. This method is unsatisfactory from the standpoint that it fails to convey to the decision-maker the uncertainty implicit in the estimate. Another approach might utilize an interval estimate rather than a point estimate. This approach has the advantage that it does convey some impression of the uncertainty involved; however, like the point estimate approach, it fails to fully utilize all the information available to the decision-maker.

An approach which does utilize available information and, at the same time, conveys the fundamental idea of uncertainty is to treat the enemy's utility function as a random variable - more specifically, as a random measure.

This assumption is not as heroic as it may first seem. It is similar to the assumption implicit in the approach of the statistician who utilizes the Bayes Principle (Ref 8:30-31) as a means of "making explicit" a decision-maker's uncertainty.

Another assumption, implicit in the definition, is that the enemy's utility function is a measure. This involves two other assumptions. First, we are assuming that the utility function is additive. Second, we are assuming the existence of an origin for measuring value. Both of these assumptions are subject to considerable criticism (Ref 9, 29) and should be viewed accordingly. We note that if this assumption is completely unacceptable one might investigate other random set functions such as those which are almost surely super-additive.

To see how such a construction benefits the decision-maker it suffices to note that by considering the enemy's utility function as a random measure the decision-maker is able to appeal to the theory of random measures for techniques that can assist him in modeling. For example, in the context of a counter-force engagement, it might be reasonable to assume that the enemy's random utility measure is homogeneous with respect to the measure which counts the number of troops in a target area (i.e., the value of a target depends only on the number of troops it contains). Similarly the enemy's random utility measure might be assumed to be stationary with respect to target location (i.e., the value of a target is independent of its location).

There are numerous other applications of the theory of random

measures. The point process is particularly useful. Agnew (Ref 2) and Goldman (Ref 12) indicate several other applications of this class of random measures.

Selected Bibliography

1. Aczél, J. Lectures on Functiona' ations and Their Applications. New York: Academic Press, 1966.

2. Agnew, R. A. Transformations of Uniform and Stationary Point Processes. Unpublished doctoral dissertation. Illinois: Northwestern University, 1968.

3. Bochner, S. "Stochastic Processes." Ann. Math., 48:1014-1061 (1947).

4. -----. Harmonic Analysis and the Theory of Probability. Los Angeles: University of California Press, 1955.

5. Choquet, G. and J. Deny. "Sur l'équation de convolution." C. R. Acad. Sci. Paris, 250:799-801 (1960).

6. Feller, W. An Introduction to Probability and Its Applications, Vol. I (Second Edition). New York: John Wiley, 1957.

7. -----. An Introduction to Probability and Its Applications, Vol. II. New York: John Wiley, 1966.

8. Ferguson, T. S. Mathematical Statistics: A Decision Theoretic Approach. New York: Academic Press, 1967.

9. Fishburn, P. C. "Utility Theory." Management Science, 14:335-381 (1968).

10. Fortet, R. M. "Random Distributions with an Application to Telephone Engineering." Proc. Third Berkley Symposium Math. Statist. Prob., 2:81-88 (1956).

11. Goldman, J. R. "Stochastic Point Processes: Limit Theorems." Ann. Math. Statist., 38:771-779 (1967).

12. -----. Infinitely Divisible Point Processes in R^n. Technical Report No. 6. Cambridge: Harvard University, 1966.

13. Halmos, P. R. Measure Theory. New Jersey: VanNostrand, 1950.

14. Harris, T. E. The Theory of Branching Processes. Berlin: Springer-Verlag, 1963.

15. -----. "Counting Measures, Monotone Random Set Functions." Z. Wahrscheinlichkeitstheorie Verw. Geb., 10:102-119 (1968).

16. Khintchine, A. Y. Mathematical Methods in the Theory of Queueing, London: Griffin, 1960.

17. Kingman, J. F. C. "Completely Random Measures." Pac. J. of Math., 21:59-78 (1967).

18. Kolmogorov, A. N. Foundations of the Theory of Probability. New York: Chelsea, 1956.

19. LeCam, L. "Un instrument d'etude des fonctions aleatoires; la fonctionelle caracteristique." C. R. Acad. Sci., Paris, 224:710-711 (1947).

20. -----. "A Stochastic Description of Precipitation." Proc. Fourth Berkley Symposium Math. Statist. Prob., 3:165-186 (1961).

21. Lee, P. M. "Infinitely Divisible Stochastic Processes." Z. Wahrscheinlichkeitstheorie Verw. Geb., 9:147-160 (1967).

22. -----. "Some Examples of Infinitely Divisible Point Processes." Studia Scientiarum Mathematicarum Hungarica, 3:219-224 (1968).

23. Mecke, J. "Stationäre Zufällige Maße auf lokalkompakten Abelschen Gruppen." Z. Wahrscheinlichkeitstheorie Verw. Geb., 9:36-58 (1967).

24. -----. "Eine Charakteristische Eigenschaft der doppelt stochastichen Poissonschen Prozesse." Z. Wahrscheinlichkeitstheorie Verw. Geb., 11:74-81 (1968).

25. Moyal, J. E. "The General Theory of Population Processes." Acta Math., 108:1-31 (1962).

26. Nawrotzki, K. "Ein Grenzwertsatz für homogene zufällige Punktfolgen." Math. Nachr., 24:201-217 (1962).

27. Neveu, J. Mathematical Foundations of the Calculus of Probability. San Francisco: Holden-Day, 1965.

28. Ryll-Nardzewski, C. "Remarks on Processes of Calls." Proc. Fourth Berkley Symposium Math. Statist. Prob., 2:455-465 (1961)

29. Samuelson, P. Foundations of Economic Analysis. Cambridge: Harvard University Press, 1963.

30. Whittle, P. "On the Variance of Yield Variance with Plot Size." Biometrika, 43:337-343 (1956).

Vita

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13. ABSTRACT
A random measure may be thought of as a random set function which is almost surely a measure. Some results obtained by Ryll-Nardzewski for point processes on the real line are extended and the Laplace functional is introduced. Completely random measures, infinitely divisible random measures, and stationary random measures are characterized. Homogeneous random measures are introduced with examples and interpretations. A general characterization theorem for homogeneous random measures is proved. Finally, several applications of the theory of random measures are given.

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